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On the Role of Vector Supersymmetry in Topological Field Theory[1]

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Abstract

Vector supersymmetry is typical of topological field theory. Its role in the construction of gauge invariant quantities is explained, as well as its role in the cancellation of the ultraviolet divergences. The example of the Chern-Simons theory in three dimensions is treated in details.

1 Introduction

A main characteristics of topological gauge theories (see [1] for a general review) is their lack of local observables. In particular there is no energy-momentum tensor because of their metric independence and, moreover, the gauge field configurations satisfying the equations of motion are pure gauges. Observables are integrated gauge invariant objects like Wilson loops, or currents localized on the boundary, if any, of the base manifold. Other important physical, gauge invariant quantities are the possible anomalies and the counterterms which correspond to the possible renormalization of the coupling constants. We shall deal here with the latter category of gauge invariant objects, which are represented by space-time integrals.

The aim of this talk is to show how a supersymmetry generated by a space-time vector valued supercharge happens to hold, and to show that this supersymmetry generator may be used in a a natural way for the explicit construction of the gauge invariant objects considered. Its role in the complete cancellation of the ultraviolet divergences in topological theories will also be explained.

In order to keep the argument as simple and self-contained as possible, I shall concentrate on the 3-dimensional Chern-Simons theory. Sect. 2 will be devoted to the description of the model, of its gauge fixing and to its BRS invariance. Sect. 3 will present some simple facts about the BRS cohomology which will be needed later on. Vector supersymmetry will be introduced in Sect. 4 and the solution of the problem in Sect.5. Some conclusions are proposed at the end.

The notations are those of [2]. Generally the statements are given without proof. The proofs may be found in [2], unless they are well-established or an explicit reference is given.

2 The $D = 3$ Chern-Simons model

2.1 The Chern-Simons action

The Chern-Simons term[3, 4] in three space-time dimensions is given by the gauge invariant action

$$S_{\mathrm{CS}} := -\frac{k}{4\pi}\int d^3x \; \varepsilon^{\mu\nu\rho}\mathrm{Tr}\left(A_\mu\partial_\nu A_\rho - \frac{2i}{3}A_\mu A_\nu A_\rho\right) \; . \tag{2.1}$$

The number k plays the role of the inverse of the coupling constant[3] and A_μ is a Lie algebra valued gauge field, as well as all the fields we shall introduce later on. Denoting such a generic field by φ, we define

$$\varphi(x) := \varphi^a(x)\tau_a \; , \tag{2.2}$$

[3]The parameter k is actually quantized[4], but this is irrelevant in perturbation theory, which is a (formal) expansion at zero coupling, hence at infinite k.

where the matrices τ are the generators of the group and obey

$$[\tau_a, \tau_b] = i f_{abc} \tau_c \ , \qquad \mathrm{Tr}\,(\tau_a \tau_b) = \delta_{ab} \ . \tag{2.3}$$

The gauge transformations read

$$\delta A_\mu(x) = \partial_\mu \omega(x) + i[\omega(x), A_\mu(x)] =: D_\mu \omega(x) \ , \tag{2.4}$$

with $\omega := \omega^a \tau_a$. The gauge group is chosen to be simple and compact. These transformations change the integrand of the action (2.1) into a total derivative, leaving thus the action invariant if there are no boundary contributions and if the topology is trivial [4].

Expression (2.1), although referred here to the flat Euclidean space-time, has an intrinsic geometrical meaning and can be defined on an arbitrary three dimensional manifold $\mathcal{E}$ in a natural way. Indeed one remarks that the Chern-Simons action (2.1) being the integral of a 3-form, does not depend on the metric $g_{\mu\nu}$ which one may introduce on $\mathcal{E}$. Explicitly, defining the gauge connection 1-form

$$A := A_\mu dx^\mu, \tag{2.5}$$

one can write the Chern-Simons action as

$$S_{\mathrm{CS}} = -\frac{k}{4\pi} \int_{\mathcal{E}} \mathrm{Tr} \left(A\, dA - \frac{2i}{3} A^3 \right) \ , \tag{2.6}$$

the wedge symbol $\wedge$ for the multiplication of forms being omitted.

The main consequence of the metric independence is the vanishing of the energy-momentum tensor associated to the Chern-Simons action:

$$T^{\mu\nu}_{\mathrm{CS}} = \frac{\delta S_{\mathrm{CS}}}{\delta g_{\mu\nu}} = 0 \ . \tag{2.7}$$

Another important feature of the theory is the peculiar form of the gauge field equations:

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] = 0 \ , \tag{2.8}$$

which means that there is also no field strength. Thus there is no local gauge invariant quantity at all, in other words there are no local observables. But this remark does not imply that the present theory is physically empty. Two types of observables may be defined. The first one is provided by the Wilson loops, that is by the gauge invariant quantities associated to the closed loops C:

$$\mathrm{Tr\ P\ exp}\, i \oint_C A_\mu dx^\mu \ , \tag{2.9}$$

where the symbol P means the "path ordered product". The second type of observables are present in case the 3-manifold possesses a 2-dimensional boundary : there exists then a set of conserved chiral current which are localized on the boundary and which generate an infinite dimensional algebra of conserved charges [5, 6, 7].

2.2 Gauge Fixing

We shall fix the gauge à la Landau, adding to the Chern-Simons action (2.1) the term

$$S_{\rm gf} := {\rm Tr} \int d^3x \ \left(B\partial^\mu A_\mu - \bar{c}\partial^\mu(\partial_\mu c + i[c, A_\mu]) \right) \ , \tag{2.10}$$

where the Lie algebra valued fields B, c and $\bar{c}$ are a Lagrange multiplier, the ghost and the antighost, respectively.

The gauge-fixed action

$$\tilde{S} := S_{\rm CS} + S_{\rm gf} \ , \tag{2.11}$$

is invariant under the nilpotent BRS transformations

$$\begin{aligned} &sA_\mu = D_\mu c := \partial_\mu c + i[c, A_\mu] \ , \qquad sc = ic^2 \ , \\ &s\bar{c} = B \ , \qquad sB = 0 \ . \end{aligned} \tag{2.12}$$

The gauge fixing part of the action of course depends on the metric, chosen here as the flat Euclidean one[4] $\delta_{\mu\nu}$.

Remark. The gauge fixing part of the action is s-exact, i.e. it is a BRS variation:

$$S_{\rm gf} = s{\rm Tr} \int d^3x \ \bar{c}\partial^\mu A_\mu \ . \tag{2.13}$$

2.3 The Slavnov-Taylor Identity

The BRS variations of most of the fields being nonlinear, one has to couple them to external fields in order to control their possible renormalizations. We thus add to the action the piece

$$S_{\rm ext} := \int d^3x \ {\rm Tr} \ \left(\rho^\mu D_\mu c + i\sigma c^2 \right) \ , \tag{2.14}$$

so that the total classical action

$$S(A, c, \bar{c}, B, \rho, \sigma) := \tilde{S} + S_{\rm ext} \tag{2.15}$$

obeys the Slavnov-Taylor identity

$$\mathcal{S}(S) := \int d^3x \ {\rm Tr} \ \left(\frac{\delta S}{\delta \rho^\mu}\frac{\delta S}{\delta A_\mu} + \frac{\delta S}{\delta \sigma}\frac{\delta S}{\delta c} + B\frac{\delta S}{\delta \bar{c}} \right) = 0 \tag{2.16}$$

which plays the role of the Ward identity associated to the BRS invariance.

[4]See [8] for the generalization to a curved manifold.

Remark. Here, at the classical level, the Slavnov-Taylor identity simply expresses the BRS invariance of the action. At the quantum level, the classical action is replaced by the vertex functional[5]

$$\Gamma(A, c, \bar{c}, B, \rho\sigma) = S(A, c, \bar{c}, B, \rho\sigma) + O(\hbar) \ , \tag{2.17}$$

i.e. by the generating functional of the 1-particule irreducible (1PI) Green functions:

$$\left.\frac{\delta^n \Gamma}{\delta\varphi(x_1)\cdots\delta\varphi(x_N)}\right|_{\varphi=0} = \langle 0|T(\varphi(x_1)\cdots\varphi(x_N))|0\rangle_{1PI} \ ,$$

where φ stands for any field. Then the Slavnov-Taylor identity (2.16) has to hold with S replaced by Γ.

The Slavnov-Taylor identity thus expresses the BRS invariance in a functional form or, equivalently, in the form of identities between Green functions.

The gauge fixing condition may also be expressed in a functional form:

$$\frac{\delta S}{\delta B} = \partial A \ , \tag{2.18}$$

Applying the functional operator $\delta/\delta B$ to the Slavnov-Taylor operator defined by (2.16) yields the identity (true for any functional $\mathcal{F}$)

$$\frac{\delta}{\delta B}\mathcal{S}(\mathcal{F}) = \mathcal{S}_{\mathcal{F}}\frac{\delta \mathcal{F}}{\delta B} + \frac{\delta \mathcal{F}}{\delta \bar{c}} \ , \tag{2.19}$$

where we have defined the "linearized" Slavnov-Taylor operator

$$\mathcal{S}_{\mathcal{F}} := \mathrm{Tr}\int d^3x \, \left(\mathrm{Tr}\,\frac{\delta \mathcal{F}}{\delta\rho^\mu}\frac{\delta}{\delta A_\mu} + \frac{\delta \mathcal{F}}{\delta A_\mu}\frac{\delta}{\delta\rho^\mu} + \frac{\delta \mathcal{F}}{\delta\sigma}\frac{\delta}{\delta c} + \frac{\delta \mathcal{F}}{\delta c}\frac{\delta}{\delta\sigma} + B\frac{\delta}{\delta\bar{c}}\right) \ . \tag{2.20}$$

Applying the identity (2.19) for $\mathcal{F} = S$ and making use of the Slavnov-Taylor identity(2.16) and of the gauge condition (2.18) yields the "ghost equation"

$$\mathcal{G}\Gamma = 0 \ , \qquad \text{with} \quad \mathcal{G} := \frac{\delta}{\delta\bar{c}} + \partial^\mu\frac{\delta}{\delta\rho^\mu} \ . \tag{2.21}$$

The latter implies that the antighost $\bar{c}$ and the external field ρ appear only in the combination

$$\hat{\rho}^\mu = \rho^\mu + \partial^\mu\bar{c} \ . \tag{2.22}$$

[5]In perturbation theory considered as a formal power expansion in $\hbar$, i.e. in the number of loops of the Feynman graphs, the zeroth order – the classical theory – coincides with the tree graph approximation.

2.4 Stability

The linearized Slavnov-Taylor operator (2.20) may be interpreted as the "derivative at the point $\mathcal{F}$" of the nonlinear map $\mathcal{S}$, defined by (2.16), from the space of the field functionals into the complex numbers. For $\mathcal{F}$ equal to the action S, it can be shown to be nilpotent as a consequence the Slavnov-Taylor identity:

$$\mathcal{S}_S{}^2 = 0 \ , \tag{2.23}$$

and its action on the fields reads

$$\begin{aligned} &\mathcal{S}_S\varphi = s\varphi \ , \quad \varphi = A, c, \bar{c}, B \ , \\ &\mathcal{S}_S\rho^\mu = \frac{\delta S}{\delta A_\mu} \ , \\ &\mathcal{S}_S\sigma = \frac{\delta S}{\delta c} \ . \end{aligned} \tag{2.24}$$

One sees that it coincides with the BRS operator s when applied to the dynamical fields, and yields equations of motion if applied to the external fields.

The operator $\mathcal{S}_S$ allows one to characterize the stability of the theory defined by the Slavnov-Taylor identity and by the gauge condition. This means the following. Given an action S obeying the equations (2.16) and (2.18), let us ask for the most general solution S' in the neighbourhood of S. Writing

$$S' = S + \varepsilon\Delta \ , \tag{2.25}$$

with ε "small", we find that the perturbation Δ has to obey the two conditions

$$\mathcal{S}_S\Delta = 0 \tag{2.26}$$

and

$$\frac{\delta\Delta}{\delta B} = 0 \ . \tag{2.27}$$

Due to the ghost equation (2.21), it must of course obey the further condition

$$\mathcal{G}\Delta = 0 \ , \tag{2.28}$$

which however is not independent due to the commutation rule

$$\left[\frac{\delta}{\delta B}, \mathcal{S}_S\right] = \mathcal{G} \ . \tag{2.29}$$

A model given by an action S is "stable" if the most general perturbation Δ can be obtained by a redefinition of the parameters and of the field variables.

It is very important to distinguish the "physical" perturbations from the "unphysical" ones. The latter perturbations are those which correspond to a mere redefinition of the fields:

they indeed don't affect the physical outcome of a theory, e.g. the scattering amplitudes in a usual gauge theory. It turns out that the unphysical $\mathcal{S}_S$-invariant perturbations are those that take the form of a $\mathcal{S}_S$-variation:

$$\Delta_{\rm unphys} = \mathcal{S}_S \hat{\Delta} \ . \tag{2.30}$$

Perturbations corresponding to the redefinition of gauge parameters are unphysical, too, and have the same form.

On the other hand, the physical perturbations correspond to the redefinition of physical parameters like the coupling constants and the masses. They are characterized by the property:

$$\mathcal{S}_S \Delta_{\rm phys} = 0 \ , \quad \text{but} \ \ \Delta_{\rm phys} \neq \mathcal{S}_S \hat{\Delta} \ \ \text{for any} \ \ \hat{\Delta} \ . \tag{2.31}$$

Solving (2.26) with a nilpotent operator $\mathcal{S}_S$ is thus a problem of cohomology. The next section will deal with this.

3 Cohomology and Descent Equations

3.1 Cohomology and Observables

In ordinary gauge theories the observables are defined, at the classical level, as gauge invariant local polynomial of the *physical fields* (gauge and matter fields). "Local" from now on will mean that these polynomials are made of products of the fields and of their derivatives at a common space-time point x. They may integrated (e.g. Wilson loops) or not (e.g. currents). In the quantum theory, defined e.g. by the renormalized perturbation theory, local field polynomials P become local "operators", or, in the framework of the Green functions, local "insertions" which may be represented by the generating functional

$$P \cdot \Gamma = P + O(\hbar) \tag{3.1}$$

of the 1-particule irreducible Green functions

$$\langle 0| T\left(P\ \varphi(x_1)\varphi(x_2)\cdots\right)|0\rangle_{1PI} \ .$$

The right-hand-side of (3.1) expresses the fact that the tree graph approximation corresponds to the classical approximation. But now P may depend on unphysical fields such as the ghost fields and the external fields.

The quantum version of an observable is then defined as an insertion given by a quantum extension of a local polynomial $\mathcal{O}$ of the *physical and unphysical fields*, which is BRS-invariant but not a BRS variation (the linearized Slavnov-Taylor operator (2.20) for $\mathcal{F} = \Sigma$ must be used):

$$\mathcal{S}_S \mathcal{O} = 0 \ , \quad \text{but} \ \ \mathcal{O} \neq \mathcal{S}_S \hat{\mathcal{O}} \ \ \text{for any} \ \ \hat{\mathcal{O}} \ . \tag{3.2}$$

In other terms, quantum observables are defined as cohomology classes of the "coboundary operator" $\mathcal{S}_S$ in the space of the local functionals.

We have seen that topological gauge theories such as the Chern-Simons theory do not possess x-dependent local observables. But they possess quantities which are gauge invariant in the sense defined above, i.e. defined by the cohomology of $\mathcal{S}$. We have already encountered the example of the nontrivial BRS-invariant perturbations of the action in Subsect. 2.4. Another important example is that of the gauge anomalies[6].

3.2 The Descent Equations

The structure we shall discuss here applies to the gauge invariant quantities which are expressed as space-time integrals. But, in order to keep simplicity, we shall consider in more details only the case of the perturbation Δ introduced in Subsect. 2.4. With

$$\Delta = \int d^3x \, Q^0(x) \ , \tag{3.3}$$

It follows from the BRS invariance condition (2.26) that the BRS variation of the integrand must be a total derivative [7]:

$$\mathcal{S}_S Q^0 = \partial_\mu Q^{1\mu} \ . \tag{3.4}$$

The conditions (2.27) and (2.28) imply that Δ – as well as all the quantities which we shall derive from it – do not depend on the Lagrange multiplier B and depend on the antighost $\bar{c}$ and the external field ρ^μ only through the combination (2.22). The relevant variables are therefore the fields A_μ, c, $\hat{\rho}^\mu$ and σ.

Applying $\mathcal{S}_S$ to (3.4) and using the nilpotency of $\mathcal{S}_S$ now imply that the variation of $Q^{1\mu}$ is the total derivative of an antisymmetric tensor:

$$\mathcal{S}_S Q^{1\mu} = \partial_\nu Q^{2[\mu\nu]} \ . \tag{3.5}$$

Repeating the argument twice leads to

$$\mathcal{S}_S Q^{2[\mu\nu]} = \partial_\rho Q^{3[\mu\nu\rho]} \ , \quad \mathcal{S}_S Q^{3[\mu\nu\rho]} = 0 \ . \tag{3.6}$$

The process stops here because the rank of an antisymmetric tensor is bounded by the space-time dimension.

In the notation of differential forms, these "descent equations" read

$$\begin{aligned}
\mathcal{S}_S \omega_3^0 &= d\omega_2^1 \ , \\
\mathcal{S}_S \omega_2^1 &= d\omega_1^2 \ , \\
\mathcal{S}_S \omega_1^2 &= d\omega_0^3 \ , \\
\mathcal{S}_S \omega_0^3 &= 0 \ ,
\end{aligned} \tag{3.7}$$

[6] Their absence in the Chern-Simons theory was actually shown in ref. [9].

[7] The exponents give the ghost number. By convention the ghost number of c is equal to 1.

with

$$\Delta = \int \omega_3^0 \ . \tag{3.8}$$

Here, d is the exterior derivative (with $d^2 = 0$), and the forms ω_{3-g}^g of ghost number g and degree $3 - g$ are defined by:

$$\begin{aligned}
\omega_3^0 &= \frac{1}{3!}\varepsilon_{\mu\nu\rho} Q^0 dx^\mu dx^\nu dx^\rho \ , \\
\omega_2^1 &= \frac{1}{2}\varepsilon_{\mu\nu\rho} Q^{1\rho} dx^\mu dx^\nu \ , \\
\omega_1^2 &= \frac{1}{2}\varepsilon_{\mu\nu\rho} Q^{2\nu\rho} dx^\mu \ , \\
\omega_0^3 &= \frac{1}{3!}\varepsilon_{\mu\nu\rho} Q^{3\mu\nu\rho} \ .
\end{aligned} \tag{3.9}$$

Remark. The formalism of the differential forms is the natural one in the present context of a topological theory. Indeed, (3.8) as well as the descent equations (3.7) never involve the metric, but only the differentiable structure of the manifold. To the contrary the metric enters explicitly, although in a spurious way, in the notation in terms of the tensors Q, the latter being the Hodge duals of the forms ω.

3.3 Solution: Existence and Uniqueness

In order to solve the descent equations, let us begin by the last and simplest of them. It is clear that, since ω_0^3 is a 0-form of ghost number 3, its most general invariant expression is given by

$$\omega_0^3 = x\frac{i}{3}\mathrm{Tr}\, c^3 \ , \tag{3.10}$$

with x an arbitrary coefficient. In order to find the solution for the other forms, and in particular for ω_3^0 yielding the quantity Δ (3.8) of interest, one has to climb up the descent equations. Each step represents a cohomology problem, but for nonintegrated objects. This cohomology can be shown to consists only of invariant 0-forms made with the ghost field c alone, without derivative[8]. This result was just used above in order to solve the last descent equation. However, since the next steps involve forms of higher degree, the cohomology is then trivial. As it can be shown, this implies that, once the 0-form ω_0^3, i.e. the coefficient x, is given, the general solution for the 3-form ω_3^0 is *unique* up to the BRS variation of a an arbitrary 3-form and the exterior derivative of an arbitrary 2-form:

$$\omega_3^0 = \bar\omega_3^0 + \mathcal{S}_S\hat\omega_3^{-1} + d\hat\omega_2^0 \ , \tag{3.11}$$

where $\bar\omega_3^0$ is a particular solution. We will show in the next section how a supersymmetry generated by a vector operator indeed gives a particular solution, and what this implies.

[8]This is very peculiar to the topological theories. In ordinary Yang-Mills theories the cohomology depends on the invariantsdraft constructed with the ghost c and on those made with the Yang-Mills strength F as well [10, 11, 12].

4 The Vector Supersymmetry

Since the action depends on the metric only through the gauge fixing part (2.10), which is a BRS variation, the energy-momentum tensor will be a BRS variation as well:

$$T^{\mu\nu} = \frac{\delta S_{\rm gf}}{\delta g_{\mu\nu}} = \mathcal{S}_S Q^{\mu\nu} \ . \tag{4.1}$$

Although the conservation of $T^{\mu\nu}$ does not strictly imply that of $Q^{\mu\nu}$, it turns out [13] that $Q^{\mu\nu}$ is indeed conserved[9]. It follows that there exists a conserved vector charge Q_μ of ghost number -1. This charge generates infinitesimal "vector supersymmetry" transformations, displayed here for all the fields in regard with their BRS transformations (see (2.24)):

$$\begin{aligned}
&\mathcal{S}_S c = ic^2 && Q_\mu c = -A_\mu \\
&\mathcal{S}_S A_\nu = D_\nu c && Q_\mu A_\nu = \frac{2\pi}{k}\varepsilon_{\mu\nu\rho}\hat\rho^\rho \\
&\mathcal{S}_S \hat\rho^\nu = -\frac{k}{4\pi}\varepsilon^{\nu\rho\sigma}F_{\rho\sigma} + i\{c,\hbar^\nu\} && Q_\mu \hat\rho^\nu = -\delta^\nu_\mu \sigma \\
&\mathcal{S}_S \sigma = D_\nu\hat\rho^\nu + i[c,\sigma] && Q_\mu\sigma = 0 \\
&\mathcal{S}_S \bar c = B && Q_\mu \bar c = 0 \\
&\mathcal{S}_S B = 0 && Q_\mu B = -\partial_\mu \bar c \ .
\end{aligned} \tag{4.2}$$

The variation of the action under these supersymmetry transformations reads

$$\begin{aligned}
\mathcal{W}^{\rm S}_{(\xi)} S &:= \int \sum_{\text{all fields } \varphi} \delta^{\rm S}_\xi \varphi \frac{S}{\delta\varphi} \\
&= \Delta^{\rm class}_\mu \ := \ \int \mathrm{Tr}\left(\frac{2\pi}{k}\varepsilon_{\mu\nu\rho}\rho^\nu\partial^\rho B + \rho^\tau\partial_\mu A_\tau - \sigma\partial_\mu c\right) \ ,
\end{aligned} \tag{4.3}$$

where we have denoted the infinitesimal supersymmetry transformation of parameter ξ^μ by

$$\delta^{\rm S}_\xi := \xi^\mu Q_\mu \ . \tag{4.4}$$

We see that the action is actually invariant for vanishing external fields ρ and σ. For the quantum theory, where the action S is replaced by the functional Γ, (4.3) is interpretated as a (broken)Ward identity for vector supersymmetry. The breaking $\Delta^{\rm class}_\mu$ vanishes when the external fields are set to zero. Moreover, being *linear* in the quantized fields, it does not need to be renormalized – whence the superscript "class". It is therefore harmless[10].

The geometrical aspect of the supersymmetry transformation laws is much more apparent if one write them – for all the fields except $\bar c$ and B – in the language of differential forms.

[9] This actually happens in every topological model studied up to now.

[10] This would not be the case for an arbitrary gauge fixing. One can check that, at least among the family of the linear covariant gauges, vector supersymmetry selects the Landau gauge. In particular a Feynman-type gauge would violate supersymmetry. Some noncovariant gauges, like the axial gauge, are also possible [14, 15, 16].

Introducing the forms

$$\tilde{\rho} = \frac{1}{2}\varepsilon_{\mu\nu\rho}\hat{\rho}^\mu dx^\nu dx^\rho \ , \quad \tilde{\sigma} = \frac{1}{3!}\sigma dx^\mu dx^\nu dx^\rho, \tag{4.5}$$

(the form A being already defined by (2.5)), and denoting by i_ξ the interior derivative, or contraction, with respect to the the vector ξ^μ, one can write, in an obviously metric independent way,

$$\begin{array}{ll} \mathcal{S}_S c = ic^2 & \delta^{\rm S}_\xi c = -i_\xi A \\ \mathcal{S}_S A = dc + i\{c, A\} & \delta^{\rm S}_\xi A = -\dfrac{2\pi}{k} i_\xi \tilde{\rho} \\ \mathcal{S}_S \tilde{\rho} = \dfrac{k}{4\pi}(dA + iA^2) + i\{c, \tilde{\rho}\} & \delta^{\rm S}_\xi \tilde{\rho} = -i_\xi \tilde{\sigma} \\ \mathcal{S}_S \tilde{\sigma} = d\tilde{\rho} + i\{A, \tilde{\rho}\} + i\{c, \tilde{\sigma}\} & \delta^{\rm S}_\xi \tilde{\sigma} = 0 \ . \end{array} \tag{4.6}$$

Now, with the help of the properties

$$d^2 = 0 \ , \qquad \{d, i_\xi\} = \mathcal{L}_\xi \ , \qquad \{i_\xi, i_{\xi'}\} = 0 \ , \tag{4.7}$$

where $\mathcal{L}_\xi$ denotes the Lie derivative along the vector ξ, it is easy to check that the algebra

$$\mathcal{S}_S = 0 \ , \quad \left\{\mathcal{S}_S, \mathcal{W}^{\rm S}_{(\xi)}\right\} = \mathcal{W}^{\rm T}_{(\xi)} \ , \quad \left\{\mathcal{W}^{\rm S}_{(\xi)}, \mathcal{W}^{\rm S}_{(\xi')}\right\} = 0 \ , \tag{4.8}$$

holds, where

$$\mathcal{W}^{\rm T}_{(\xi)} := \int \sum_{\text{all fields } \varphi} \mathcal{L}_\xi \varphi \frac{\delta}{\delta\varphi} \tag{4.9}$$

is the generator of the space-time translations[11]. This shows that the BRS, supersymmetry and translation generators $\mathcal{S}_S$, $\mathcal{W}^{\rm S}_{(\xi)}$ and $\mathcal{W}^{\rm T}_{(\xi)}$ obey a superalgebra of the Wess-Zumino type.

Let us end this section by noting that the supersymmetry Ward identity (4.3) imposed to the theory yields the constraint

$$\mathcal{W}^{\rm S}_{(\xi)}\Delta = 0 \tag{4.10}$$

on the perturbation (2.25), which has to be obeyed together with the previous constraints (2.26), (2.27) and (2.28).

5 Solving the Descent Equations

We have mentioned in Subsect. 3.3 that the descent equations (3.7) admit a unique solution – modulo a total derivative and a BRS variation – once the bottom 0-form is given. We shall

[11]In the flat space and with the constant vector fields considered here, this Lie derivative takes the simple form of an infinitesimal translation $\xi^\mu\partial_\mu$. In a general curved manifold $\mathcal{L}_\xi$ generates the diffeomorphisms along the vector field ξ, and supersymmetry transformations become superdiffeomorphisms [8].

show in the present section, on the example of the equations (3.7), how vector supersymmetry allows to find that solution.

This is done with the help of the operator[12]

$$\nabla := dx^\mu \mathcal{W}_\mu \ , \tag{5.2}$$

where we have introduced the components $\mathcal{W}^{\rm S}_\mu$ of the supersymmetry generator (4.3) – and also $\mathcal{W}^{\rm T}_\mu$ for the translations – defined by

$$\mathcal{W}^{\rm S}_{(\xi)} =: \xi^\mu \mathcal{W}^{\rm S}_\mu \ , \quad \mathcal{W}^{\rm T}_{(\xi)} =: \xi^\mu \mathcal{W}^{\rm T}_\mu \ . \tag{5.3}$$

It is clear from the algebra (4.8) which now reads

$${\mathcal{S}_S}^2 = 0 \ , \quad \left\{ \mathcal{S}_S, \mathcal{W}^{\rm S}_\mu \right\} = \mathcal{W}^{\rm T}_\mu \ , \quad \left\{ \mathcal{W}^{\rm S}_\mu, \mathcal{W}^{\rm S}_\nu \right\} = 0 \ , \tag{5.4}$$

that

$$\prod_{k=1}^{n} \mathcal{W}^{\rm S}_{\mu_k} = 0 \quad \text{ if } n \geq 3 \ . \tag{5.5}$$

Hence

$$\nabla^n = 0 \quad \text{ if } n \geq 3 \ . \tag{5.6}$$

The algebra (4.8) also implies the commutation relation[13]

$$[\nabla, \mathcal{S}_S] = d \ . \tag{5.7}$$

The particular solution of the descent equations we are looking for reads

$$\omega^{3-p}_p = \frac{1}{p!} \nabla^p \omega^3_0 \ , \quad p = 0, \cdots, 3 \ . \tag{5.8}$$

This is readily checked using the commutation relation (5.7). The corresponding perturbation Δ (3.8) of the action then reads

$$\Delta = \frac{1}{3!} \int \nabla^3 \omega^3_0 = ix \int \mathrm{Tr} \left(\frac{k}{6\pi} A^3 + \tilde\sigma c^2 + \tilde\rho \{A, c\} \right) \ . \tag{5.9}$$

This is only one of the BRS invariant solutions corresponding to the same zero 0-form ω^3_0, as we already know. Another possible solution is e.g. the one which leads to the Chern-Simons action (2.1):

$$\Delta = x \Sigma_{\rm CS} \ . \tag{5.10}$$

[12]A more intrinsic definition of ∇, suitable for the generalizations, is provided by its action on the individual fields (in the differential form notation):

$$\nabla c = -A \ , \quad \nabla A = -\tfrac{2\pi}{k} \tilde\rho \ , \quad \nabla \tilde\rho = -\tilde\sigma \ , \quad \nabla \tilde\sigma = 0 \ . \tag{5.1}$$

[13]This is Sorella's "decomposition of the exterior derivative" [17].

(One can actually check that (5.9) and (5.10) differ by a BRS variation.) The solution (5.10) would correspond to a renormalization of the parameter k. However the Chern-Simons action $\Sigma_{\rm CS}$ is not supersymmetric and one has thus to discard this solution.

On the other hand, the solution (5.9) being obtained by the triple application of the supersymmetry operator is supersymmetric due to the identity (5.5) taken for $n = 4$. Moreover, it is the only supersymmetric one, as we shall check below. But before, let us mention that, like (5.10), it also corresponds to a renormalization of k, however combined with a renormalization of the field amplitudes.

Let us write the general solution, which differs from (5.9) by an arbitrary BRS variation (restricted by having the same quantum numbers and dimension as the action):

$$\begin{aligned}\Delta_{\rm general} &= \Delta + \mathcal{S}_S \int {\rm Tr}\ (y\tilde{\sigma}c + z\tilde{\rho}A) \\ &= \Delta + \int {\rm Tr}\ \left(y(i\tilde{\sigma}c^2 + \tilde{\rho}dc + i\tilde{\rho}\{A,c\}) + z\left(\frac{k}{4\pi}(dAA + iA^3) - \tilde{\rho}dc\right)\right)\ .\end{aligned}$$

But the variational part in the right-hand side is not supersymmetric for any ratio of the parameters y and z: this shows that indeed (5.9) is the unique perturbation of the action which is both BRS invariant and supersymmetric.

6 Ultraviolet Finiteness

6.1 Classical Perturbation

Gauge theories quantized in the Landau gauge are characterized by a remarkable property, which may be stated in words as the nonrenormalization of the insertions depending on the ghost field c not differentiated. This property is expressed by the "antighost equation" [18] which, in the present case reads

$$\overline{\mathcal{G}}S := \int d^3x\ \left(-i\frac{\delta}{\delta c} + \left[\bar{c}, \frac{\delta}{\delta B}\right]\right) S = \Delta^G_{\rm class}\ :=\ \int d^3x\ \left([\rho^\mu, A_\mu] - [\sigma, c]\right)\ , \tag{6.1}$$

where the breaking $\Delta^G_{\rm class}$, like the one of the supersymmetry Ward identity, is linear in the quantum fields, hence not subject to renormalization. For the perturbation Δ defined in Subsect. 2.4, this leads to the further constraint

$$\int d^3x\ \frac{\delta}{\delta c}\Delta = 0\ . \tag{6.2}$$

Now, it is visible that (5.9) violates this latter constraint, and thus must be discarded. As a final result, taking into account all the constraints dictated by the initial theory on the perturbation Δ leads to the solution

$$\Delta = 0\ . \tag{6.3}$$

6.2 Counterterms Induced by the Radiative Corrections

Up to now, we have spoken only of the classical theory, and more specifically of its stability. The outcome is that the classical theory is not only stable: its parameters themselves are completely fixed.

Going to the quantum theory, in the perturbative framework, requires the investigation of two points.

1) Absence of anomalies

One has to check the validity to all orders of the Slavnov-Taylor identity (2.16), of the supersymmetry Ward identities (4.3), and as well of the constraints given by the gauge condition, the ghost equation and the antighost equation. In other words one has to check that all these identities do not suffer from anomalies. It has actually been verified [9, 13, 2] that algebraic consistency forbids the appearance of such anomalies.

2) Absence of counterterms

One has to check the stability of the theory upon the radiative corrections: these corrections must be interpretable at each order as a renormalization of the parameters of the theory. For this one has to study all possible counterterms to the action. The counterterms are of course constrained by the symmetries and identities imposed to the Green functions. It is a very general feature of renormalization theory [2] that these constraints on the counterterms are identical to the constraints on the perturbation of the classical action we have already spoken about. We can therefore retain our previous result (6.3). This simply means the absence of any free counterterm – hence of any free parameter once those of the classical theory have been fixed. The usual interpretation in the framework of a renormalization via a cut-off procedure is the ultraviolet finiteness of the theory. One can however offer a more physical interpretation [13], namely that the quantum theory keeps intact the *scale invariance* of the original classical theory.

7 Conclusion

The role of vector supersymmetry, as we have seen, is twofold.

Its first role is to realize explicitly Sorella's decomposition of the external derivative (5.7), thus providing the operator ∇ for the construction of solutions to the descent equations.

The second aspect of this supersymmetry is its role in the cancellation mechanism of the ultraviolet divergences. In fact, as we have seen in the Chern-Simons example, super-

symmetry alone is not sufficient. The special coupling of the ghost c as expressed by the "antighost equation" (6.1) is also needed. One actually observes a "conspiration" between both conditions: supersymmetry selects the unique element (5.9) among the whole family of BRS invariant counterterms, and it is just this element which is killed by the the condition (6.2) following from the antighost equation.

Many of the results shown here for the three-dimensional Chern-Simons theory have been generalized to a whole class of topological theories, including the BF models [19, 20, 21, 22], the bosonic string [23, 24], four-dimensional topological Yang-Mills theory [25, 26], etc.

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